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Counting Derangements, Non Bijective Functions and the Birthday Problem[1]

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Summary. The article provides counting derangements of finite sets and counting non bijective functions. We provide a recursive formula for the number of derangements of a finite set, together with an explicit formula involving the number e. We count the number of non-one-to-one functions between to finite sets and perform a computation to give explicitely a formalization of the birthday problem. The article is an extension of [10].

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The articles [13], [16], [9], [1], [4], [7], [5], [6], [14], [2], [8], [3], [11], [12], [17], [18], and [15] provide the notation and terminology for this paper.

1. Preliminaries

In this paper x denotes a set.

Let us note that every finite 0-sequence of $\mathbb{Z}$ is integer-valued.

Let n be a natural number. Observe that $n!$ is natural.

Let n be a natural number. One can verify that $n!$ is positive.

Let c be a real number. Observe that $\exp c$ is positive.

Let us observe that e is positive.

One can prove the following propositions:

(1) $\mathrm{id}_{\emptyset}$ has no fixpoint.

(2) For every real number c such that $c < 0$ holds $\exp c < 1$.

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2. Rounding

Let n be a real number. The functor $\operatorname{round} n$ yields an integer and is defined as follows:

(Def. 1) $\operatorname{round} n = \lfloor n + \frac{1}{2} \rfloor$.

We now state two propositions:

(3) For every integer a holds $\operatorname{round} a = a$.

(4) For every integer a and for every real number b such that $|a - b| < \frac{1}{2}$ holds $a = \operatorname{round} b$.

3. Counting Derangements

The following propositions are true:

(5) Let n be a natural number and a, b be real numbers. Suppose $a < b$. Then there exists a real number c such that $c \in \,]a, b[$ and $\exp a = (\sum_{\alpha=0}^{\kappa} (\text{Taylor}(\text{the function exp}, \Omega_{\mathbb{R}}, b, a))(\alpha))_{\kappa \in \mathbb{N}}(n) + \frac{\exp c \cdot (a-b)^{n+1}}{(n+1)!}$.

(6) For every positive natural number n and for every real number c such that $c < 0$ holds $|-n! \cdot \frac{\exp c \cdot (-1)^{n+1}}{(n+1)!}| < \frac{1}{2}$.

Let s be a set. The functor $\operatorname{derangements} s$ is defined by:

(Def. 2) $\operatorname{derangements} s = \{f; f$ ranges over permutations of s: f has no fixpoint$\}$.

Let s be a finite set. Observe that $\operatorname{derangements} s$ is finite.

One can prove the following propositions:

(7) Let s be a finite set. Then $\operatorname{derangements} s = \{h : s \to s$: h is one-to-one $\wedge \bigwedge_x (x \in s \Rightarrow h(x) \neq x)\}$.

(8) For every non empty finite set s there exists a real number c such that $c \in \,]-1, 0[$ and $\overline{\overline{\operatorname{derangements} s}} - \frac{\overline{\overline{s}}!}{e} = -\overline{\overline{s}}! \cdot \frac{\exp c \cdot (-1)^{\overline{\overline{s}}+1}}{(\overline{\overline{s}}+1)!}$.

(9) For every non empty finite set s holds $|\overline{\overline{\operatorname{derangements} s}} - \frac{\overline{\overline{s}}!}{e}| < \frac{1}{2}$.

(10) For every non empty finite set s holds $\overline{\overline{\operatorname{derangements} s}} = \operatorname{round}(\frac{\overline{\overline{s}}!}{e})$.

(11) $\operatorname{derangements} \emptyset = \{\emptyset\}$.

(12) $\operatorname{derangements}\{x\} = \emptyset$.

The function the der seq from $\mathbb{N}$ into $\mathbb{Z}$ is defined by the conditions (Def. 3).

(Def. 3)(i) (The der seq)$(0) = 1$,

(ii) (the der seq)$(1) = 0$, and

(iii) for every natural number n holds (the der seq)$(n + 2) = (n + 1) \cdot (($the der seq$)(n) + ($the der seq$)(n + 1))$.

Let c be an integer and let F be a finite 0-sequence of $\mathbb{Z}$. Observe that $c\,F$ is finite, integer-valued, and transfinite sequence-like.

Let c be a complex number and let F be an empty function. Note that $c\,F$ is empty.

We now state three propositions:

(13) For every finite 0-sequence F of $\mathbb{Z}$ and for every integer c holds $c \cdot \sum F = \sum((c\,F){\restriction}(\operatorname{len} F -' 1)) + c \cdot F(\operatorname{len} F -' 1)$.

(14) Let X, N be finite 0-sequences of $\mathbb{Z}$. Suppose $\operatorname{len} N = \operatorname{len} X + 1$. Let c be an integer. If $N {\restriction} \operatorname{len} X = c\,X$, then $\sum N = c \cdot \sum X + N(\operatorname{len} X)$.

(15) For every finite set s holds (the der seq)($\overline{\overline{s}}$) = $\overline{\overline{\text{derangements}\, s}}$.

4. Counting not-one-to-one Functions and the Birthday Problem

Let s, t be sets. The functor not-one-to-one(s,t) yielding a subset of t^s is defined as follows:

(Def. 4) not-one-to-one$(s,t) = \{f : s \to t$: f is not one-to-one$\}$.

Let s, t be finite sets. Observe that not-one-to-one(s,t) is finite.

The scheme *FraenkelDiff* deals with sets $\mathcal{A}$, $\mathcal{B}$ and a unary predicate $\mathcal{P}$, and states that:

$$\{f : \mathcal{A} \to \mathcal{B} : \text{not } \mathcal{P}[f]\} = \mathcal{B}^{\mathcal{A}} \setminus \{f : \mathcal{A} \to \mathcal{B} : \mathcal{P}[f]\}$$

provided the following condition is met:

- If $\mathcal{B} = \emptyset$, then $\mathcal{A} = \emptyset$.

Next we state three propositions:

(16) For all finite sets s, t such that $\overline{\overline{s}} \leq \overline{\overline{t}}$ holds $\overline{\overline{\text{not-one-to-one}(s,t)}} = \overline{\overline{t}}^{\overline{\overline{s}}} - \frac{\overline{\overline{t}}!}{(\overline{\overline{t}} -' \overline{\overline{s}})!}$.

(17) For every finite set s and for every non empty finite set t such that $\overline{\overline{s}} = 23$ and $\overline{\overline{t}} = 365$ holds $2 \cdot \overline{\overline{\text{not-one-to-one}(s,t)}} > \overline{\overline{t^s}}$.

(18) For all non empty finite sets s, t such that $\overline{\overline{s}} = 23$ and $\overline{\overline{t}} = 365$ holds $\mathrm{P}(\text{not-one-to-one}(s,t)) > \frac{1}{2}$.

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